Exercise 3

Use power series to solve the differential equation.

$$y' = x^2 y$$

Solution

$x = 0$ is an ordinary point, so the ODE has a power series solution centered here.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate the series with respect to x.

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substitute these formulas into the ODE.

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = x^2 \sum_{n=0}^{\infty} a_n x^n$$

Bring x inside the summand.

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^{n+2}$$

Make the substitution $n = k + 3$ in the series on the left and the substitution $n = k$ in the series on the right.

$$\sum_{k+3=1}^{\infty} (k+3) a_{k+3} x^{(k+3)-1} = \sum_{k=0}^{\infty} a_k x^{k+2}$$

Simplify the sum on the left.

$$\sum_{k=-2}^{\infty} (k+3) a_{k+3} x^{k+2} = \sum_{k=0}^{\infty} a_k x^{k+2}$$

Write out the first two terms on the left.

$$(-2+3) a_{-2+3} x^{-2+2} + (-1+3) a_{-1+3} x^{-1+2} + \sum_{k=0}^{\infty} (k+3) a_{k+3} x^{k+2} = \sum_{k=0}^{\infty} a_k x^{k+2}$$

Combine the series on the left side.

$$a_1 + 2a_2 x + \sum_{k=0}^{\infty} \left[(k+3) a_{k+3} - a_k\right] x^{k+2} = 0$$

In order for the left side to be zero, a_1 and a_2 and the quantity in square brackets must be zero.

$$(k+3) a_{k+3} - a_k = 0$$

Solve for a_{k+3}.

$$a_{k+3} = \frac{1}{k+3}a_k \qquad a_1 = 0 \qquad a_2 = 0$$

In order to determine a_k, plug in values for k and try to find a pattern.

$$\begin{aligned}
k = 0: &\quad a_3 = \frac{1}{0+3}a_0 = \frac{1}{3}a_0 \\
k = 1: &\quad a_4 = \frac{1}{1+3}a_1 = 0 \\
k = 2: &\quad a_5 = \frac{1}{2+3}a_2 = 0 \\
k = 3: &\quad a_6 = \frac{1}{3+3}a_3 = \frac{1}{6}\left(\frac{1}{3}a_0\right) = \frac{1\cdot 1}{6\cdot 3}a_0 \\
k = 4: &\quad a_7 = \frac{1}{4+3}a_4 = 0 \\
k = 5: &\quad a_8 = \frac{1}{5+3}a_5 = 0 \\
k = 6: &\quad a_9 = \frac{1}{6+3}a_6 = \frac{1}{9}\left(\frac{1\cdot 1}{6\cdot 3}a_0\right) = \frac{1\cdot 1\cdot 1}{9\cdot 6\cdot 3}a_0 \\
&\quad \vdots
\end{aligned}$$

The general formulas are

$$\begin{aligned}
a_{3m} &= \frac{1}{(3m)!!!}a_0 = \frac{1}{3^m m!}a_0 \\
a_{3m+1} &= 0 \\
a_{3m+2} &= 0.
\end{aligned}$$

Therefore, the general solution is

$$\begin{aligned}
y(x) &= \sum_{m=0}^{\infty} a_m x^m \\
&= \sum_{m=0}^{\infty} a_{3m}x^{3m} + \sum_{m=0}^{\infty} a_{3m+1}x^{3m+1} + \sum_{m=0}^{\infty} a_{3m+2}x^{3m+2} \\
&= \sum_{m=0}^{\infty} \frac{1}{3^m m!}a_0 x^{3m} + \sum_{m=0}^{\infty} (0)x^{3m+1} + \sum_{m=0}^{\infty} (0)x^{3m+2} \\
&= a_0 \sum_{m=0}^{\infty} \frac{x^{3m}}{3^m m!} \\
&= a_0 \sum_{m=0}^{\infty} \frac{\left(\frac{x^3}{3}\right)^m}{m!} = a_0 e^{x^3/3},
\end{aligned}$$

where a_0 is an arbitrary constant.